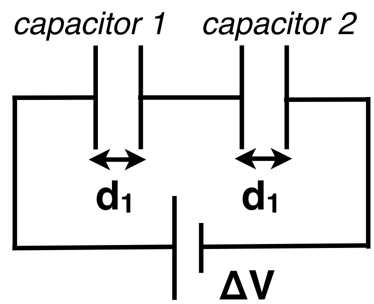
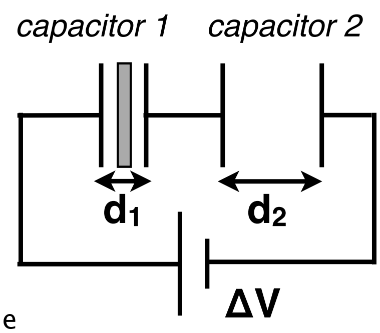
7. Capacitor Networks – II.



Two air-filled capacitors of plate area **A** and plate separation $\mathbf{d_1}$ are connected to a battery, as depicted to the right. For the purposes of this question, take the permittivity of air to be equal to $\boldsymbol{\varepsilon_o}$, the permittivity of free space.

a) Derive the equivalent capacitance of the circuit depicted in the upper diagram using **A**, $\mathbf{d_1}$ and $\boldsymbol{\varepsilon_o}$



b) Pokium has a dielectric constant of $\mathbf{K_p}$. A slab of pokium with a thickness of $\mathbf{d_1/3}$ and an area **A** (the same area as the capacitor plates) is placed in capacitor 1, as depicted to the right. Capacitor 1 now has a series of $\mathbf{d_1/3}$ of air, then $\mathbf{d_1/3}$ of pokium, then $\mathbf{d_1/3}$ of air between its plates. Find the equivalent capacitance of this new circuit using **A**, $\mathbf{d_1}$, $\mathbf{K_p}$, $\boldsymbol{\varepsilon_o}$ and $\mathbf{d_2}$, where $\mathbf{d_2}$ is the plate separation for capacitor 2.

c) $\mathbf{K_p}$ = 2.0 is the dielectric constant of pokium. If the two depicted circuits were set up to have the same equivalent capacitance, and $\mathbf{d_1}$ = 3.0cm, what would be the value of $\mathbf{d_2}$, the plate separation for capacitor 2, in cm?

d) $\boldsymbol{\varepsilon_o} = 8.9 \times 10^{-12}$ F/m, **A** = 10.0 cm^2, and the battery supplies $\mathbf{\Delta V}$ = 12 V. In the final configuration of the circuit depicted in the lower diagram, for $\mathbf{d_2}$ as calculated in part **(c)** of this problem, what is the stored energy in capacitor 2?

<u>Hint:</u> note that the equivalent capacitance was set up to be the same for the two depicted circuits.

(a) THE EQUATION FOR CAPACITANCE BASED ON THE PHYSICAL CHARACTERISTICS OF A CAPACITOR IS $C = \frac{k\varepsilon_0 A}{d}$. NEITHER CAPACITOR CONTAINS A DIELECTRIC SO $k=1$ FOR BOTH CAPACITORS AND THUS $C = \frac{\varepsilon_0 A}{d_1}$ FOR EACH CAPACITOR.

THE EQUATION FOR EQUIVALENT CAPACITANCE FOR A SERIES CIRCUIT IS $\frac{1}{C_{EQ}} = \frac{1}{C_1} + \frac{1}{C_2}$

SO $\underline{\underline{\frac{1}{C_{EQ}}}} = \frac{d_1}{\varepsilon_0 A} + \frac{d_1}{\varepsilon_0 A} = \underline{\underline{\frac{2d_1}{\varepsilon_0 A}}}$. THIS COULD BE SIMPLIFIED FURTHER, BUT THERE'S NO NEED AS WE'VE DONE WHAT THE PROBLEM ASKED.

(b) CAPACITOR 1 CONTAINS A SERIES OF AIR, POKIUM AND AIR <u>IN SERIES</u> EACH IS $\frac{d_1}{3}$ THICK AND OF AREA A. ONE HAS $k = k_p$ AND TWO HAVE $k=1$. THIS SERIES WILL ACT LIKE THREE CAPACITORS IN SERIES, SO THE EQUIVALENT CAPACITANCE FOR CAPACITOR 1 IS $\frac{1}{C_1} = \frac{d_1}{3\varepsilon_0 A} + \frac{d_1}{3k_p\varepsilon_0 A} + \frac{d_1}{3\varepsilon_0 A} = \frac{2d_1}{3\varepsilon_0 A} + \frac{d_1}{3k_p\varepsilon_0 A}$

CAPACITOR 2 IS AS FOR PART (a) BUT WITH SEPARATION $d_2 \Rightarrow \frac{1}{C_2} = \frac{d_2}{\varepsilon_0 A}$

$$\underline{\underline{\frac{1}{C_{EQ}}}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{2d_1}{3\varepsilon_0 A} + \frac{d_1}{3k_p\varepsilon_0 A} + \frac{d_2}{\varepsilon_0 A} = \underline{\underline{\frac{2d_1}{\varepsilon_0 A}\left(\frac{1}{3} + \frac{1}{6k_p} + \frac{d_2}{2d_1}\right)}}$$

AGAIN, ANY REARRANGE ANSWERS THE QUESTION

(c) WE ARE ASKED TO FIND d_2 WHEN C_{EQ} FOR THE FIRST CIRCUIT $= C'_{EQ}$ FOR THE SECOND CIRCUIT. WHEN $C_{EQ} = C'_{EQ}$, $\frac{1}{C_{EQ}} = \frac{1}{C'_{EQ}}$. SO IT'S EASIEST TO EQUATE THE $\frac{1}{C_{EQ}}$ EXPRESSIONS THAT WE'VE DERIVED. $\frac{1}{C_{EQ}} = \frac{1}{C'_{EQ}} \Rightarrow \frac{\cancel{2d_1}}{\cancel{\varepsilon_0 A}} = \frac{\cancel{2d_1}}{\cancel{\varepsilon_0 A}}\left(\frac{1}{3} + \frac{1}{6k_p} + \frac{d_2}{2d_1}\right) \Rightarrow 1 = \frac{1}{3} + \frac{1}{6k_p} + \frac{d_2}{2d_1}$

$\Rightarrow \frac{d_2}{2d_1} = \frac{2}{3} - \frac{1}{6k_p} \Rightarrow d_2 = \frac{4d_1}{3} - \frac{d_1}{3k_p} = \frac{d_1}{3}\left(4 - \frac{1}{k_p}\right)$ OR SOME SIMILAR REARRANGE

SUBSTITUTING IN FOR $k_p = 2.0$ AND $d_1 = 3.0\,cm$, $\underline{\underline{d_2}} = \frac{\cancel{3.0}\,cm}{\cancel{3}}\left(4 - \frac{1}{2.0}\right) = \underline{\underline{3.5\,cm}}$

(d) WE NEED THE STORED ENERGY IN CAPACITOR 2. AS CHARGE IS THE SAME ACROSS ALL ELEMENTS IN A SERIES CIRCUIT, THE EASIEST WAY TO PROCEED IS TO FIND THE CHARGE ACROSS THE WHOLE CIRCUIT (ONE WAY TO THINK ABOUT THIS IS THE "EQUIVALENT CHARGE"). ONCE WE KNOW THIS CHARGE, WE KNOW THE CHARGE ON CAPACITOR 2 AS $Q_{EQ} = Q_1 = Q_2$

WE ARE GIVEN $U = \frac{1}{2}CV^2$ FOR THE STORED ENERGY AND $C = \frac{Q}{V}$ DEFINES THE CAPACITANCE. SO, $U = \frac{1}{2}CV^2 = \frac{1}{2}C\left(\frac{Q}{C}\right)^2$ $\Rightarrow U = \frac{1}{2}\frac{Q^2}{C}$. SO FOR CAPACITOR 2 $U_2 = \frac{1}{2}\frac{Q_2^2}{C_2} = \frac{1}{2}\frac{Q_{EQ}^2}{C_2}$ (WHERE WE USED $Q_1 = Q_2 = Q_{EQ}$)

ALL THAT IS LEFT IS TO FIND C_2 AND Q_{EQ}. AS FOR PART (a) AND (b) $\frac{1}{C_2} = \frac{d_2}{\varepsilon_0 A}$. WE CAN FIND Q_{EQ} FROM $C_{EQ} = \frac{Q_{EQ}}{V_{EQ}}$ (WHERE WE'RE USING "EQ" TO MEAN "ACROSS THE ENTIRE CIRCUIT"). WE KNOW A SIMPLE EXPRESSION FOR C_{EQ} FROM PART (a) AND WE'RE GIVEN A HELPFUL HINT TO REMIND US $C_{EQ} = C'_{EQ} = \frac{\varepsilon_0 A}{2d_1}$ SUBSTITUTING IN FOR EVERYTHING:

$$U_2 = \frac{1}{2}\frac{Q_{EQ}^2}{C_2} = \frac{1}{2}\frac{C_{EQ}^2 V_{EQ}^2}{C_2} = \frac{V_{EQ}^2}{2}\frac{\varepsilon_0^2 A^2}{4d_1^2}\frac{d_2}{\varepsilon_0 A} = \frac{V_{EQ}^2}{8}\varepsilon_0 A \frac{d_2}{d_1^2}$$

$\varepsilon_0 = 8.9 \times 10^{-12}$ F/m, $d_1 = 3.0$ cm, $A = 10\text{ cm}^2 = 10^{-3}\text{ m}^2$, $V_{EQ} = 12$V, AND WE FOUND $d_2 = 3.5$cm IN PART (c). SUBSTITUTING IN, $U_2 = \frac{144}{8} 8.9 \times 10^{-12}\, 10^{-3} \frac{3.5 \times 10^{-2}}{(3.0 \times 10^{-2})^2}$
$\sim \underline{\underline{6.2 \times 10^{-12}\text{ J}}}$